Looking for Structure: Moving Out of the Realm of Computation to Explore the Nature of the Operations

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Although K-6 mathematics focuses on calculation procedures for adding, subtracting, multiplying, and dividing, the meaning of the operations may fall into the background. Indeed, many common errors stem from confusion about which structures apply to which operation. Early algebra, with its emphasis on recognizing and expressing mathematical structure, has the potential of rooting out such errors. This paper presents a teaching model designed to bring students' attention to the unique structures associated with each operation as they notice, articulate, illustrate, and prove generalizations that contrast the behaviour of different operations.

Third-grade teacher, Alice Kaye, posts the following pairs of equations for her class to examine and asks her students what they notice about the problems. (Alice Kaye is a pseudonym, as are the names of her students.)

7 + 5 = 12 7 + 6 = __	7 + 5 = 12 8 + 5 = __
9 + 4 = 13 9 + 5 = __	9 + 4 = 13 10+ 4 = __
What do you notice? What's happening here?	

Ms. Kaye acknowledges to the students that the numbers are not challenging for them. The purpose of the discussion is not to solve the problems, but to consider what is going on between the pairs of equations and to state their ideas clearly and convincingly. The class fills in the blanks, and then begins to talk about what they notice. Students mention that one number changes, one stays the same, and the last number in the equation changes, too. The discussion goes on like this for a few minutes, until one child, Evan, says, "Since 9 + 4 is 13, 9 + 5 *has* to be 1 more than 13."

This discussion launches for the class an exploration of generalizations about the behavior of the operations. In this case, students are investigating what happens to the sum when one addend is increased by 1. They will work together to state a conjecture, represent it, and justify why it must be true for all whole numbers. They will go on to consider what happens when analogous changes are made with other operations. Such examination of structure—which brings *operations* to the fore and leaves *numbers* in the background—is an aspect of early algebra.

Introduction

In extant practice, the teaching of calculation in the elementary and middle grades tends to focus on procedures for producing correct results. Although students realize there is a different procedure for each operation, the distinction among operations may fall into

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 28-39. Auckland: MERGA.

the background. Confusion about the operations frequently results in consistent procedural errors. Indeed, common errors in subtraction or multiplication can be interpreted as an application of structures that apply only to addition.

For example, consider such errors as these:

- 35 – 16 = 21 Decompose the numbers into tens and ones; subtract the tens (30 – 10) and subtract the ones (6 – 5); add the results (20 + 1). The correct answer is 19.
- 35 × 16 = 330 Decompose the numbers into tens and ones; multiply the tens (30 × 10) and multiply the ones (5 × 6); add the results (300 + 30). The correct answer is 560.

The basic approach behind these errors is related to a strategy that works for addition: To add 35 + 16, decompose the numbers into tens and ones; add the tens (30 + 10) and add the ones (6 + 5). The sum of the results provides the correct answer, 51. Students who make the errors illustrated above may be thinking of their correct addition strategy as the way *numbers* work, rather than how *addition* works. For that reason, despite a teacher's corrections, students often continue to apply the incorrect procedure, and the error is likely to persist unless its underlying foundation is examined.

The realization that each operation has its own set of properties and behaviors is fundamental in understanding mathematics. Implications apply not only in the elementary grades as students study whole numbers, fractions, and decimals, but also in later years to make sense of operations on new classes of numbers, such as integers or complex numbers, and on algebraic expressions. Indeed, we see confusion about the operations similar to those illustrated above carry over into errors frequently made by algebra students:

- $(a + b)^2 = a^2 + b^2$ (instead of $a^2 + 2ab + b^2$)
- $2(ab) = (2a)(2b)$ (instead of $(2a)b$)

In making these errors, students may be applying the same symbol pattern as the distributive property, ignoring how addition and multiplication must come into play.

Early algebra, with its emphasis on recognizing and expressing mathematical *structure* (Kieran, 2017 et al.), has the potential of rooting out such errors. A focus on the behavior of addition, subtraction, multiplication, and division helps students come to see an operation not exclusively as a process or algorithm, but also as a mathematical object in its own right (Kieran, 1989; Sfard, 1991; Slavitt, 1999). As the operations become salient, seen as objects with a set of characteristics unique to each, students are positioned to recognize and apply their distinct structures.

Key to this work is encouraging students to notice and express regularities in the number system and to verbalize in general terms the strategies they apply in their calculations. As students move to a level of generalization, they engage in metacognitive acts (Cusi et al., 2010). In the language of Malara and Navarra, students "substitute the act of calculating with looking at oneself while calculating" (p. 2002, p. 230).

Also key to this work are representations—diagrams, physical objects, or story contexts—that embody relationships among quantities defined by the operations (Warren & Cooper, 2009; Moss & McNab, 2011). For example, addition may be represented as the joining of two sets and subtraction as comparison or removal. An arrangement of equal groups or an array can represent multiplication or division. Such representations are used to explain how and why general strategies must work. As illustrated in the classroom examples below, images of the operations become thinking tools for students that they can call upon to reason about arithmetic symbols.

Phases of an Investigation

Over the last fifteen years, I have been working with a team of researchers and teacher collaborators to investigate the learning of students when teachers focus attention on the structure of the operations (Russell et al, 2011; Schifter, 2011, 2018; Schifter et al, 2007). As part of our research, we created a teaching model comprised of five phases of investigation (Russell et al, 2017).

I. Noticing regularity. Students examine pairs or sequences of related problems, equations, or expressions and describe patterns they notice.
II. Articulating a claim. Based on the patterns noticed in phase I, students work individually and collectively to write a conjecture clearly enough so that someone not in the class could understand it.
III. Investigating through representations. Students represent specific instances of the claim with manipulatives, diagrams, and story problems.
IV. Constructing arguments. Students extend their representations to explain why the conjecture must be true for all (whole) numbers.
V. Comparing and contrasting operations. Once a claim has been verified, students consider the question, Does this work for other operations? This leads them back to phase I, to consider a set of problems, equations, or expressions that illustrate an analogous claim that could be made for another operation, and then to go on to phases II to IV, eventually to prove a second conjecture.

Phase I: Noticing Regularity

As illustrated in the opening of this paper, an investigation begins with students examining a set of related problems, equations, or expressions and describing what they see. When Evan offered his insight, “Since 9 + 4 is 13, 9 + 5 *has* to be 1 more than 13,” Ms. Kaye recognized the importance of Evan’s statement with its assertion of necessity and asked the class to break into pairs to discuss it. When the class gathered for further discussion, his classmate, Pamela, said, “I was just wondering. How did Evan come up with the idea he had? Because these are not just everyday ideas that you come up with every day.”

Evan responded, “I’m not really sure. I just know it. It kind of seems obvious to me, so I didn’t think to think about it before.”

Most classrooms are likely to include both students like Evan for whom such generalizations are so obvious as to be invisible—they never realized there was something to pay attention to—and students like Pamela, who had never looked for such patterns before. Pamela and many of her classmates were coming to see that noticing patterns in the number system is an important kind of mathematical activity.

Whether an individual student is for the first time finding patterns in the number system, is searching for language to describe the pattern he or she has noticed, or is beginning to articulate the mathematical relationships that underlie the pattern, all students in the class are engaged in the same task. The entire class benefits from the range of questions and ideas, although students may each learn something different from the very same discussion.

Phase II: Articulating a Claim

Noticing regularity leads to articulating a claim that a relationship holds for a set of numbers. As students begin to explore a generalization, there is a tendency to declare that *it* works, without specifying what *it* is. The class may appear to agree, but it isn't necessarily clear that they are talking about the same idea. Thus, in the teaching model, students are challenged to come up with language to state the generalization they have noticed.

Toward the end of the lesson introduced above, Ms. Kaye asked students to work individually or in pairs to either write a statement that described what was going on or come up with more examples of the same phenomenon. The next day, she posted additional numerical examples along with the following student statements.

- In the first column, if the number goes up, the answer goes up.
- The first number goes up by 1 and the second number stays the same, so the last number goes up by 1.
- One number grows by 1, so the sum grows by 1.

Among the goals of the lesson is that students learn the language of generalization. Yet, the objective is not to provide students with the most precise and concise statement of a special case of the associative property of addition. (Represented in algebraic notation, the class is working on: $a + (b + 1) = (a + b) + 1$.) Rather, students use their own language and work together to create a statement that is clear enough that someone outside the class would understand. This is one aspect of what Malara and Navarra (2002) call "algebraic babbling." Analogous to the way children learn natural language, students learn to communicate in algebraic language by starting from its meaning and through collective discussion, verbalization, and argumentation, gradually become proficient in syntax.

Note that in the students' statements above, the first two have no mention of the operation involved; in the third statement, the word *sum* implies addition. As mentioned above, as students begin this work, numbers tend to have greater salience and the operations fall into the background. In facilitating the discussion in which the students collaborate to produce a "class conjecture," the teacher points out aspects of their statements that need attention, such as specifying that two numbers are added.

The teacher may also introduce technical vocabulary and symbols as the need arises, especially to clarify referents. For example, if students use the word "number" to refer to different objects in their claim, the teacher might ask students how someone reading their words will know what they're referring to. She might suggest such terms as "addend" or "sum" so that students hear more precise language and begin to use it themselves.

By the end of Ms. Kaye's lesson, the class agreed on the following statement as their conjecture: *In addition, if you increase one of the addends by 1 and keep the other addend the same, then the sum will also increase by 1.*

Phase III: Representing

In the elementary grades, representations in the form of physical models, drawings, diagrams, number lines, arrays, and story contexts can be tools for reasoning, communicating, and constructing arguments. Representations that embody the relationships defined by the operations allow students to examine *why* the symbol patterns they identified work. They help students develop their own internal logic and connect the words of their conjecture to images of the operation as well as its symbolic representation.

In Ms. Kaye's class, students worked in pairs to create representations that illustrate the relationships in the generalization being explored. One such representation is shown in Figure 1, in which the action and meaning of the operation of addition is represented as the joining of stacks of cubes. There are several additions: the joining of the light gray and dark gray stacks; the joining of the single white cube to the dark gray stack; and the joining of the single white cube to the light gray and dark gray stacks.

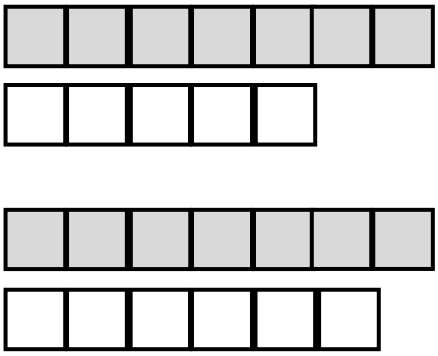

Figure 1. A representation of the class conjecture.

When the class came together to share their work, Ms. Kaye prompted the discussion of each representation by asking a set of questions designed to help students make connections to the elements of their claim. At this point, the discussion referred to specific numbers: 7 + 5 = 12 and 7 + 6 = 13.

- Where do you see 7 and 5?
- Where do you see the sum?
- How is addition represented?
- Where do you see the 1 that was added to 5?
- How does your representation show the sum increased by 1?

Although the discussion still refers to specific numbers, these questions emphasize how the representation shows the operation. The questions may seem straightforward, but they take students into deeper understanding of the representation and often uncover confusions the class needs to work through about the mathematical relationships they are attempting to represent. As they answer the same questions for a variety of representations, students recognize what is common across all of them and see how the same relationship can appear in different forms.

Phase IV: Constructing an Argument

The notion of proof is new to most elementary students, and some may not understand the purpose of it. The goal at this phase of the learning is to deepen students' understanding of the conjecture, to have them consider what it means to make a claim for an infinite class of numbers, and to engage in discussion of the representations that could be used to prove the conjecture.

For example, Andrew, who created the representation shown in Figure 1, explained that the stacks of cubes could stand for any numbers. "This number [points to the first row of light gray cubes] plus this number [second row of dark gray cubes] equals the sum. Then … the same number here [third row of light gray cubes] plus the same number plus one [points to the last row of dark gray cubes and the white cube] equals the sum plus one."

Andrew's explanation is independent of the number of light gray or dark gray cubes in his stacks. The representation is not just about specific quantities but can be used to show what happens with any whole numbers. Andrew makes this clear by referring to each stack as "this number" rather than naming the actual number of cubes in the stacks; that is, the light gray and dark gray stacks can represent any two whole number addends. Finally, the representation shows *why* the conjecture must be true. When the white cube is added to the dark gray stack (1 is added to an addend), in that same action the combined light and dark gray stack increases by one cube (the sum increases by 1).

Again, especially because these ideas are new to students, they work through the variety of arguments constructed by members of the class. As students compare different representations and identify commonalities, the common mathematical structure at the basis of each becomes prominent.

Phase V: Comparing and Contrasting Operations

Even with the emphasis on addition in the previous discussions, if students have not had experience thinking about the operations as objects, each with its own properties, they frequently think of generalizations as about *numbers*, rather than about an *operation.* That is, when they notice a generalization, they assume the same number patterns will hold, whether they insert the symbols, +, -, ×, or ÷. Often, after they investigate a generalization about one operation, they are surprised to discover that the same pattern does not hold for other operations. However, as soon as they try to apply the same conjecture to another operation, they may run into counterexamples and realize that, for this other operation, they must look for another regularity and state a different conjecture. Because one of the goals of this work is for students to understand more deeply the structure that is particular to each operation, it is important to explore sets of related generalizations that highlight these differences.

Ms. Kaye began this second investigation in the same way she began the first. She presented her students with pairs of equations and asked what they noticed.

<table>
<tr><td>$7 \times 5 = 35$
$7 \times 6 = 42$</td><td>$7 \times 5 = 35$
$8 \times 5 = 40$</td></tr>
<tr><td>$9 \times 4 = 36$
$9 \times 5 = 45$</td><td>$9 \times 4 = 36$
$10 \times 4 = 40$</td></tr>
<tr><td colspan="2">What do you notice?
What's happening here?</td></tr>
</table>

After some discussion, she asked students to work individually in response to this prompt: *In a multiplication problem, if you add 1 to a factor, what will happen to the product?*

Multiplication was new to the class and not all students at first noticed the regularity evident in the problems. The following are some of the statements written by students who could see the result of adding 1 to a factor:

- The number that is not increased is the number that the answer goes up by.

- The number that is staying and not going up, increases by however many it is.
- I think that the factor you increase, it goes up by the other factor.

Because some of the students were not yet able to see the regularity, Ms. Kaye decided to postpone work on a more precise statement of the claim and, instead, to have the class investigate the equations with representations. In this way, students would develop a stronger image of multiplication and could then recognize the pattern in the arithmetic symbols.

In order to scaffold the investigation, Ms. Kaye gave the following assignment.

> Choose which of the original equation pairs you want to work with. Write a story for the original equation; then change it just enough to match the new equations.
> Then do one of these:
> - Draw a picture for the original equation; then change it just enough to match the new equations.
> - Make an array for the original equation; then change it just enough to match the new equations.
>
> Example: Original equation: $7 \times 5 = 35$
> New equations: $7 \times 6 = 42$, $8 \times 5 = 40$

Students worked in pairs and then came together to share their work, each pair presenting their story problem and picture or array. Elise and Maria's story was, "There are 7 jewelry boxes with 5 jewels in each box. There are 35 jewels in all." For this, they drew a diagram that showed 7 groups of 5, as shown in Figure 2.

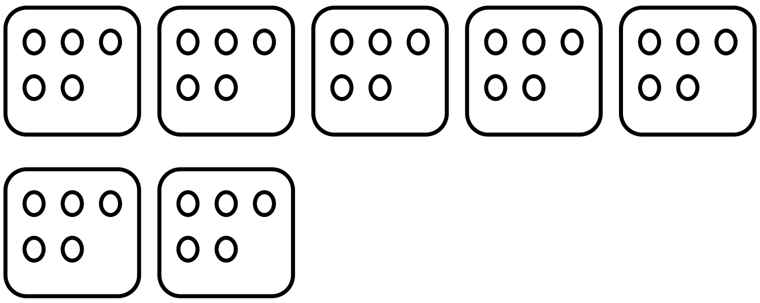

Figure 2. A representation of 7 × 5.

To change the story to 8 × 5, they said there was one more jewelry box, increasing the total by 5. They showed the new jewelry box by coloring it, as shown in Figure 3.

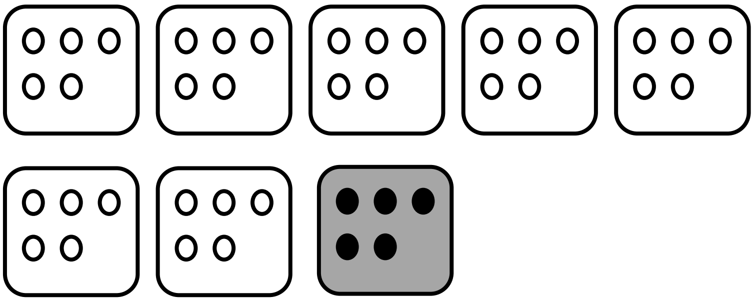

Figure 3. Comparing 7 × 5 and 8 × 5.

To change 7 × 5 to 7 × 6, the girls said that 1 more jewel was added to each box. They indicated the additional jewels in their picture by coloring a dot in each group. When 5 increased to 6, there were 7 additional jewels, one for each box, as shown in Figure 4.

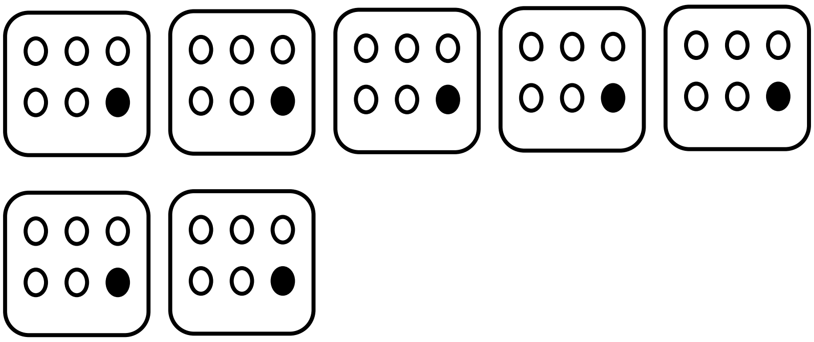

Figure 4. Comparing 7 × 5 and 7 × 6.

Many of the students had story contexts and diagrams like Elise and Maria's. Some of the students presented arrays to show what happens when you add one row or one column. With each pair's presentation, the class was provided a new representation of the concept, solidifying the ideas as they saw them expressed in different stories, different pictures, and different arrangements of cubes.

Throughout the presentations, the class worked on stating what was taking place: When 1 is added to 7, the product increases by 5, the size of one group. When 1 is added to 5, the product increases by 7 because each group increases by 1. By the end of their discussion of different representations, they had developed a more general class conjecture: *When the first factor increases by 1, the product increases by 1 group equal to the second factor. When the second factor increases by 1, each group is 1 larger, so the product increases by the number of groups, the first factor.* (In general in the United States, the first factor designates the number of groups and the second factor, the size of each group.)

Now the class had engaged in two investigations: 1) what happens to the sum when 1 is added to an addend? and 2) what happens to the product when 1 is added to a factor? However, students may remember these two investigations as distinct, unrelated activities and not necessarily reflect back on the difference between the two operations. For that reason, after students presented their arguments about multiplication, Ms. Kaye asked the class to reread their addition and multiplication conjectures and think about their arguments. "What is different between our two conjectures?"

The question prompted students to articulate how multiplication differs from addition. When adding two numbers, the addends refer to the same unit: 7 + 5 = 12 might mean 7 jewels combined with 5 jewels to make a group of 12 jewels. However, when multiplying two numbers, the factors refer to different units: 7 × 5 might mean 7 boxes each containing 5 jewels, to make 35 jewels altogether.

Impact on Student Learning: Focus on Operations versus Focus on Numbers

The design of the teaching model was built on the insights and creativity of the collaborating teachers, working with the research and development team, to investigate how to approach generalizations about the operations in elementary classrooms. Using the model as a basis, the researchers wrote eight lesson sequences (Russell et al, 2017), each consisting of about twenty 15- to 20-minute sessions, to be taught over several weeks in addition to regular math instruction. Each sequence focused on a pair of analogous generalizations for two operations, for example, how to form equivalent addition expressions versus how to form equivalent subtraction expressions, or, as illustrated in the example above, the result of adding an amount to an addend versus the result of adding an amount to a factor. Each collaborating teacher field tested two of the eight lesson sequences, one in the fall semester, the other in the spring. The generalizations to be explored depended on the grade level, second through fifth grades. The fourth- and fifth-grade sequences included generalizations that extended the domain from whole numbers to fractions or decimals.

From the project in which the lesson sequences were developed and field tested, as well as a prior project focused on related ideas, three data sources provide evidence of the impact on student learning of lessons that focus on the behavior of the operations: 1) documentation of the field tests of the lesson sequences, along with teachers' reports, 2) a set of interviews with individual students of the collaborating teachers, and 3) a written assessment of students whose teachers participated in an online professional development course.

Documentation of Field Tests and Teacher Observations

Based on observation and documentation of the field tests, staff and collaborating teachers identified four key areas of student learning: noticing regularities and making conjectures, developing mathematical language, engaging in mathematical reasoning, and recognizing that the operations have meaning, properties, and behaviors (Russell et al, 2017). Specifically, with regard to the last area, teachers reported on changes of how students engaged in their regular mathematics lessons. Many teachers wrote about how students had become curious about the number system and pursued questions they posed for themselves.

> During our regular math instruction, we spent time noticing and discussing patterns we may not have paid any attention to prior to our work [with the lesson sequences]. Students used language like, "I can prove they're related!" and "Let's write a conjecture for that!"

> Because of the emphasis on reasoning about mathematical ideas and noticing regularity [in the lesson sequences], I find students referencing that all the time. For example, while doing a page [from the student activity book], students will often write notes on the side like, "This is just like our conjecture…" Sometimes, students ask questions of each other like, "But will that always work?" Sometimes, they just have a more systematic approach to reasoning through ideas.

A fourth-grade teacher, whose class explored what happened when their conjecture about multiplication was extended from whole numbers to fractions, wrote,

> I used to feel like students learned one way of multiplying, and then I would show them how to multiply differently when they were introduced to fractions and decimals. Now students are thinking of multiplying as something that stays the same and looking for the ways it plays out with new classes of numbers.

Furthermore, teachers reported, the lesson sequences engaged the range of students. Students who were not typically the fastest, most vocal, and confident math students became central to this work.

> Students are more likely to notice patterns and to feel confident to share their ideas as well as ask questions about things they don't fully understand. The level of confidence, a freedom to speak their minds, and their ability to speak intelligently about mathematical ideas has greatly improved.

> Over the course of these sequences I have seen students more willing to share their thinking and students more willing to work harder to answer their own questions through the use of representations and manipulatives. They also become risk takers. Someone would throw out an idea and others would then add on. Sometimes the language was hard to unscramble, but they all felt confident just to contribute even if it only started the conjecture. This really helped to carry into regular math class. Students who were timid and shy seem to find their voice after we worked in the routines sessions.

There were opportunities for struggling students to continue to work on creating representations, and at the same time more advanced students pushed themselves to think more deeply about refining a conjecture or developing a proof. All of these students were strengthening their understanding of the unique mathematical structures of the operations.

Interview Data

As part of a teaching experiment with twelve field test teachers of grades 2 to 5, one-on-one interviews were conducted with three students from each classroom representing the range of learners, characterized in terms of strong, average, or weak in grade-level computation. In one strand of the interview, students were given pairs of subtraction problems (for example, 10 – 3 = 7; 10 – 4 = ?) that illustrate a structural property not explored in the instructional sequences: *Given a subtraction expression, if the second term (the subtrahend) increases by 1, the difference decreases by 1*. Students were asked to describe what they noticed, come up with other pairs of problems that illustrate the same feature, state a conjecture, and use a representation to explain why the conjecture must be true.

In the analysis of interview data (Higgins, in preparation), one of the dimensions that distinguished students' conjectures was "salience of the operation: the degree to which students attend to the behavior of the operation versus focus almost exclusively on the numbers when drawing generalizations and articulating conjectures." Some students articulated generalizations that were fundamentally about the operation: "When you have the same numbers, once you subtract more, you'll have less. And if you subtract less, you will have more." Other students showed no evidence of attention to the operation: "The numbers in the middle, you just add 1. Then the answer you take away 1. The first numbers are the same."

In the database (n = 36), in interviews conducted at the beginning of the year, lack of salience of the operation was found for close to half the students. After having worked on lesson sequences that explored a different set of generalizations about the operations, the percentages improved. At each grade level, more students explicitly referenced the operation or talked about what they were noticing in operation-specific terms. The operation was no longer just part of the background but became something that students realized they needed to attend to when articulating what they were noticing.

Written Assessment Data

In a project prior to developing the lesson sequences, the team designed an on-line professional development course (Russell et al, 2012 b) with the following goals: to help teachers understand and look for structural properties implicit in students' work in number and operations, bring students' attention to such properties, and support students to articulate, represent, and create mathematical explanations of the properties. The first year the course was offered, pre- and post-course assessment data was collected from 600 students of 36 participating teachers and, as comparison, 240 students from 16 non-participating teachers in the same school systems (Russell, et al., 2017). Items included those in which students were asked to explain why they think two expressions are equal. Student responses were coded for the type of explanation they provided: a) no explanation; b) a computational explanation; or c) a relational explanation, that is, an explanation that refers to mathematical structure. For instance, to explain why 9 – 5 and 10 – 6 are equal, a student could carry out both computations, showing that each expression equals 4, or the student could give a relational explanation: e.g., "Since 9 is 1 less than 10 and 5 is 1 less than 6, the difference is the same." In the post-intervention assessment, students of teachers in the Participant Group provided significantly more relational explanations than in the Comparison Group.

The assessment also asked students in grades 3 to 5 to write a story problem for a given multiplication expression. In the posttest, 74% of the Participant Group (n = 475) produced a correct story, but only 48% of the Comparison Group (n = 180). Students of grades K to 2 were asked to write a story problem for a subtraction expression. Although the Participant Group (n = 128) showed significant progress from pretest to posttest—from 28% correct to 74% correct—the difference with the Comparison Group (n = 60) was not significant.

Conclusion

Data from all three sources suggest that lessons in which teachers draw students' attention to the distinct structures of each operation help to make the operations a salient object in students' mathematical experience. Teachers reported that implementation of the lesson sequences shifted the culture of the regular mathematics classroom, in which students exhibited curiosity about the operations, knew how to explore their own questions, and recognized their conjectures in their calculation work. The interview data demonstrate how, when students notice patterns across calculation problems, they recognize the pattern as related to the structure of a given operation. The written assessment data reveal that, once students have an opportunity to explore and represent properties of the operations, they have a better understanding of contexts that are modeled by the operations and rely on structures to explain the equivalence of arithmetic expressions.

However, within the interview data, even after the intervention, there were still students at each grade level that produced conjectures in which the operation was invisible. In the post-intervention written assessments, there were students who continued to rely on computation to prove the equivalence of two expressions and students who could not create a story problem for a given arithmetic expression. To make the operations salient objects in *all* students' mathematical experience requires persistent effort.

Acknowledgments

The research and development described in this paper was conducted over many years in collaboration with Susan Jo Russell and Virginia Bastable; Traci Higgins and Reva Kasman more recently joined the research team. Megan Franke contributed to the analysis of the written assessment data. All of the work depended on the insight and creativity of collaborating teachers. The work was partially supported by the National Science Foundation (ESI-0550176, EHR-1019482). Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation).

References

Cusi, A., Malara, N, & Navarra, G. (2011). Theoretical issues and educational strategies for encouraging teachers to promote a linguistic and metacognitive approach to early algebra. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 483-510). New York: Springer.

Higgins, T. (in preparation). A Conceptual Framework for Analyzing Key Aspects of Student Generated Conjecture and Generalization in the Elementary Grades.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Volume 4 of *Research agenda for mathematics education*, pp. 33-56). Reston, VA: National Council of Teachers of Mathematics.

Kieran, C., Pang, J.S., Ng, S.F., Schifter, D., & Steinweg, A.S. (2017). Topic Study Group 10: Teaching and learning of early algebra. In G. Kaiser (Ed.), *The Proceedings of the 13th International Congress on Mathematical Education*. New York: Springer.

Malara, N.A., & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologna, Italy: Pitagora.

Moss, J. & McNab, S.L. (2011). An approach to geometric and numeric patterning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 277-301). New York: Springer.

Russell, S.J., Schifter, D., & Bastable, V. (2012a). *Connecting Arithmetic to Algebra: Strategies for Building Algebraic Thinking in the Elementary Grades*. Portsmouth, NH: Heinemann.

Russell, S.J., Schifter, D., & Bastable, V. (2012b). Facilitators Guide for *Connecting Arithmetic to Algebra: Strategies for Building Algebraic Thinking in the Elementary Grades*. Portsmouth, NH: Heinemann.

Russell, S.J., Schifter, D., Bastable, V., & Franke, M. (2016). Algebraic Reasoning in the Elementary Classroom: Results of a Professional Development Program for Teachers. https://www.terc.edu/display/Library/Algebraic+Reasoning+in+the+Elementary+Classroom%3A+Results+of+a+Professional+Development+Program+for+Teachers

Russell, S.J., Schifter, D., Kasman, R.. Bastable, V., & Higgins, T. (2017). *But Why Does It Work? Mathematical argument in the elementary grades.* Portsmouth, NH: Heinemann.

Schifter, D. (2011). Examining the Behavior of Operations: Noticing Early Algebraic Ideas. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics Teacher Noticing: Seeing through Teachers' Eyes* (pp. 204-220). New York: Routledge

Schifter, D. (2018). Early Algebra as Analysis of Structure: A Focus on Operations. In C. Kieran (Ed.). *Teaching and Learning Algebraic Thinking With 5- To 12-Year-Olds: The Global Evolution of an Emerging Field Of Research And Practice* (pp, 309-327). New York: Springer.

Schifter, D., Monk, G, S., Russell, S. J., & Bastable, V. (2007). Early Algebra: What Does Understanding the Laws of Arithmetic Mean in the Elementary Grades? In J. Kaput, D. Carraher, & M. Blanton (Eds.) *Algebra in the early grades* (pp. 413-447). Mahwah, NJ: Lawrence Erlbaum Associates.

Warren, E., & Cooper, T. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal*, Vol. 21, (2), 76-95.